Virtual Teaching of Linear Algebra with Complex Systems and Artificial Intelligence Case Studies

Patrik Christen and Terry Inglese, 30 July 2020
Determinants

Proof of Formula for $2 \times 2$ Matrices

Proof: $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is proved for $2 \times 2$ matrices.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, it follows from $AA^{-1} = I$

that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = I$. 
Overarching Educational Framework

Backward Design
Best Practices
Paul Lockhart’s Approach

Image: The American Go Association
The Big Picture

Application → Complex Systems, Artificial Intelligence
Language → Notation, Definitions
Appreciation → Fun, Creativity, Beauty, Complexity
Learning Objectives

1. Students understand and are able to explain basic linear algebra concepts and to solve exercises.

2. Students are able to relate these basic linear algebra concepts to complex systems and artificial intelligence case studies.

3. Students have fun and are amazed.
Instructional Concept

1. Motivation
2. Computational Exploration
3. Foundation / Case Study
4. Exercise
5. Tutorial / Exploration and Case Discussion
1) Motivation

Video and Lecture Notes

1. Vectors and Matrices → Structure of Systems
2. Systems of Linear Equations → Equilibrium of Systems
3. Linear Transformations → Transformations of Systems
4. Determinants → Characteristics of Systems
5. Eigenvalues and Eigenvectors → Dynamics of Systems
Computational Exploration

It is now time to go into the wild of mathematics and explore creatures called vectors and matrices. Think of yourself as a scientist doing some field work such as a field biologist studying an animal in the rainforest or a climate researcher studying a glacier in the Alps. Below is a field guide for the computational exploration of vectors and matrices:

a) Since this is the first mathematical field trip in these lecture notes, we first need to acquire some equipment. The device that we are going to use studying mathematical creatures is a software called MATLAB. Install MATLAB: 1) Check eligibility on https://ch.mathworks.com/academia/tah-support-program/eligibility.html 2) Confirm the email that you will receive afterwards and create a MATLAB account 3) Download MATLAB from you account and install it.

b) Equipped with MATLAB, we can now go into the field and create some vectors and matrices before we observe them in the next step. In MATLAB, you can create a vector with, e.g. \(v = [1; 3; 4]\) and matrices with, e.g. \(A = [1 2; 2 3; 1 4]\). Experiment yourself by creating your own vectors and matrices. How do they look like? How do they differ?

c) Knowing how vectors and matrices look like, we can now move on to observe their behaviour. We know what happens if we add two numbers together. Do vectors and matrices behave in the same way with regard to addition? How about multiplication?

Make notes of your experiments and come up with a description of the structure and behaviour of the two creatures vectors and matrices.
3) Foundation / Case Study

Lecture Notes

Definition 2.5.8. A diagonal matrix whose diagonal components are all equal to 1 is called an identity matrix and is denoted by $I$.

Example 2.5.8. An example of an identity matrix $I \in \mathbb{R}^{3 \times 3}$ is given below:

\[
I = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Definition 2.5.9. A matrix where all components are zero is called the zero matrix and is denoted by $O$.

Example 2.5.9. An example of a zero matrix $O \in \mathbb{R}^{3 \times 3}$ is given below:

\[
O = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

2.6 Properties of Matrix Operations

Besides basic and advanced matrix operations, matrices also have certain properties with respect to these operations. Properties of matrix addition and scalar multiplication can be derived from properties of vector operations. They are given in the following theorem:

Theorem 2.6.1. Let $A, B, C \in \mathbb{R}^{m \times n}$ and $c, d \in \mathbb{R}$. Then the properties of matrix addition and scalar multiplication are

a) $A + B = B + A$

b) $(A + B) + C = A + (B + C)$

c) $A + O = A$

d) $A + (-A) = O$

e) $c(A + B) = cA + cB$

f) $(c + d)A = cA + dA$

g) $c(dA) = (cd)A$

h) $IA = A$

Case Study

4.1 Graphs

Leonhard Euler (1707 – 1783) was born in Basel and is regarded one of the greatest mathematicians of all time. He is famous for solving all sorts of practical problems and at the same time contributed substantially to the advancement of pure mathematics. On such fundamental insight is what we call graphs today and he came up with it to solve a longstanding puzzle of the seven bridges of Königsberg. People in Königsberg (now Kaliningrad) enjoyed walking along the river Pregel and, at least according to the legend, wondered whether it would be possible to walk through the city by crossing each of the seven bridges exactly once (Fig. 4.1). We will now look at how Euler was able to solve this problem. His approach is as remarkable as his solution to the problem.

Figure 4.1

Instead of walking endlessly along the river and across bridges, Euler reformulated the problem in a more abstract manner. He thought of the problem as a network or graph, that is only considering nodes and their connections. The two islands (Fig. 4.1 A and D) and the two riverbanks (Fig. 4.1 B and C) are considered nodes or vertices.
Exercise

4.1 Foundation

a) Create concrete examples of \( u, v \in \mathbb{R}^n \) and \( c \in \mathbb{R} \) by assuming the components of \( u \) and \( v \) as well as some values for \( n \) and \( c \). Perform \( u + v \), \( c(u + v) \), \( u \cdot v \), \( \|u\| \), and \( d(u, v) \). Create more examples and perform these operations on them until you feel comfortable with vector operations and properties.
5) Tutorial / Exploration and Case Discussion

Virtual Meeting
## Course Programme

**Guided Self-Study:** Motivation, Computational Exploration, Foundation, Case Study, Exercise  
**Virtual Meeting:** Tutorial, Exploration and Case Discussion  

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Course Assessment

Graded Documentation of Exploration, Exercise, Tutorial, and Exploration and Case Discussion

Context Effort and Learning

Content Correctness and Creativity